(1) Homogeneity of geodesics

$$\left.\begin{array}{l}\frac{DX}{dt}=0 \\ p\in M\end{array}\right\}$$ then there exists an open set $V\subset M$ around p

and $\delta>0$, $\varepsilon_1>0$ ~~and~~ a map

$$E: \,]-\delta,\delta[\, \times \mathcal{U} \longrightarrow M$$

with $\mathcal{U}=\{(q,v):\ q\in V,\ v\in T_qM,\ |v|<\varepsilon_1\}$

s.t. a curve

$t\longrightarrow E(t,q,v)$ $\quad t\in\,]-\delta,\delta[$ is

unique solution of $\frac{DX}{dt}=0$ passing through q at $t=0$ and for which $v\in T_qM$

$$\begin{cases}E(0,q,v)=q\\ \dot E(0,q,v)=v\end{cases}$$

Observe that if $\underline{E(t,q,v)}$ is a sol. for $\frac{DX}{dt}=0$,
then $\underline{E(st,q,v)\text{ also is.}}$

$$\frac{D}{dt}\left(\frac{d\gamma}{dt}\right)=0 \Rightarrow \frac{D}{d(st)}\frac{d\gamma}{d(st)}=0$$

$\underline{s\neq 0}$
Moreover $E(st,q,v)=E(t,q,sv)$

$$\dot E(st,q,v)\big|_{t=0}=s\,\dot E(0,q,v)=sv$$

$$\Rightarrow \boxed{E(st,q,v)=E(t,q,sv).}$$

Using this we may make interval of definition for geodesic uniformly large.

For example $t \in]-2,2[$:

$$(*) \quad \left.\begin{array}{l} \frac{DX}{dt} = 0 \\ p \in M \end{array}\right\} \text{ there exists } V \subset M \text{ and } \varepsilon > 0 \text{ and a map}$$

$$E : \underline{\underline{]-2,2[}} \times \mathcal{U}' \longrightarrow M$$

$$\mathcal{U} = \{(q,\omega) : q \in V,\ \omega \in T_qM,\ |\omega| < \varepsilon\}$$

<u>Indeed:</u>

$E(t', q, v)$ was defined for $|t'| < \delta$ and $|v| < \varepsilon_1$

$$\begin{array}{l} \shortparallel \\ E\left(\frac{\delta t}{2}, q, v\right) = E\left(t, q, \frac{\delta}{2} v\right) \end{array}$$

is defined for $|t| < 2$ and

$$\omega = \frac{\delta}{2} v \quad \text{s.t.} \quad |\omega| < \frac{\delta \varepsilon_1}{2} = \varepsilon.$$

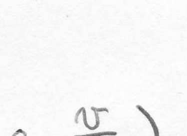
$\square$.

② <u>Exponential map</u>

$$\exp_q(v) = E(1, q, v) = E\left(|v|, q, \frac{v}{|v|}\right)$$

where $v \in \mathcal{U}$ as in $(*)$.

$v \in B_\varepsilon(0) \subset T_qM$



$$\exp_q : B_\varepsilon(0) \longrightarrow M$$

"Travel from q length equal to $|v|$ along geodesic with velocity $\frac{v}{|v|}$".

Calculate the differential of $\exp_q$:

$$d(\exp_q)_0(v) = (\exp_q)_{*0}(v) = \frac{d}{dt} \exp_q(tv)\Big|_{t=0} =$$

$$= \frac{d}{dt} E(1, q, tv)\Big|_{t=0} = \frac{d}{dt} E(t, q, v)\Big|_{t=0} = v$$

$$\Rightarrow \quad d(\exp_q)_0 = id\big|_{T_qM}$$

$\Rightarrow$ inverse function theorem says that $\exp_q$ is a diffeomorphism ~~on~~ in ~~an~~ a neighbourhood around 0 in T_qM

Prop

Given $q \in M$ there exists $\varepsilon > 0$ such that

$$\exp_q : B_\varepsilon(0) \to M \text{ is a diffeomorphism}$$

onto an open subset of M.

(3) Terminology

If $\exp_q$ is a diffeomorphism of a neighbourhood V of 0 in T_qM, $\exp_q(V)$ is called normall neighbourhood of q in M.

If $B_\varepsilon(0)$ in T_qM is such $\overline{B_\varepsilon(0)} \subset V$ then $\exp_q B_\varepsilon(0) = B_\varepsilon(q)$ is called normal ball (or geodesic ball) with center in q and radius ε.

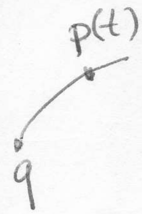
Normal coordinates in $\exp_q V \subset M$

$e = (e_\mu)$ basis in $T_q(M)$

$$e : T_q(M) \longrightarrow \mathbb{R}^n$$
$$X = X^\mu e_\mu \longmapsto (X^\mu)$$

$$x_e : \exp_q(V) \longrightarrow \mathbb{R}^n$$

$$x_e = e \circ \exp_q^{-1} : p \longrightarrow x_e(p).$$



$$p(t) = E(t, q, X) = E(1, q, tX) = \exp_q(tX)$$

$$x_e(t) = tX^\mu$$

So in normal coordinates equation for geodesics read

$$x_e(t) = tX^\mu$$

$$\Gamma^\mu_{\nu\rho}(tX^\alpha)$$

$$0 = \frac{d^2 x_e^\mu}{dt^2} + \Gamma^\mu_{\nu\rho}\, \dot{x}_e^\mu \dot{x}_e^\rho = \Gamma^\mu_{\nu\rho}\, X^\nu X^\rho$$

$$\Rightarrow \boxed{\Gamma^\mu_{\nu\rho}(q) + \Gamma^\mu_{\rho\nu}(q) = 0}$$

(4) Example Lobachevski metric

$$g = \frac{dx^2+dy^2}{y^2} = \theta^{1\,2} + \theta^{2\,2} = g_{\mu\nu}\theta^\mu\theta^\nu \qquad g_{\mu\nu} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$\theta^1 = \frac{dx}{y}, \quad \theta^2 = \frac{dy}{y}$$

$$d\theta^1 = \frac{dx\wedge dy}{y^2} = \left[\theta^1\wedge\theta^2 \right. \quad = -\cancel{\Gamma^1{}_1\theta^1} - \left.\Gamma^1{}_2\,\theta^2\right] \Rightarrow \Gamma^1{}_2 = -\theta^1 + \alpha\,\theta^2 = \Gamma_{12}$$

$$d\theta^2 = \left[0 \right. \quad = -\Gamma^2{}_1\theta^1 - \left.\cancel{\Gamma^2{}_2\theta^2}\right] \Rightarrow 0 = \Gamma_{12}\theta^1 = \alpha\,\theta^2\theta^1$$

$$\Rightarrow \alpha = 0$$

$$\Rightarrow \quad \Gamma_{ij}: \quad \text{only } \boxed{\Gamma_{12} = -\theta^1} \qquad \Gamma_{11} = \Gamma_{22} = 0$$

$$\Omega_{ij} = d\Gamma_{ij} + 0$$

$$\Rightarrow \text{only } \underbrace{\Omega_{12} = d\Gamma_{12} = -d\theta^1 = -\theta^1\wedge\theta^2}$$

$\Downarrow$ space of constant curvature

$$\underline{\underline{K = -1}} \quad !$$

Geodesics!

Tangent vector $V = V^\mu X_\mu$. The frame $(X_1, X_2) = (y\partial_x, y\partial_y)$.

The geodesic equation in this coframe is:

$$\frac{dV^\mu}{dt} + \Gamma^\mu{}_{\nu\rho}V^\nu V^\rho = 0$$

$$V^\mu = \begin{pmatrix}A\\B\end{pmatrix} \Rightarrow \left.\begin{aligned}&\frac{dA}{dt} + \Gamma^1{}_{21}BA = 0\\ &\frac{dB}{dt} + \Gamma^2{}_{11}A^2 = 0\end{aligned}\right\} \Rightarrow \left\{\begin{aligned}&\frac{dA}{dt} - AB = 0\\ &\frac{dB}{dt} + A^2 = 0\end{aligned}\right.$$

Solving:

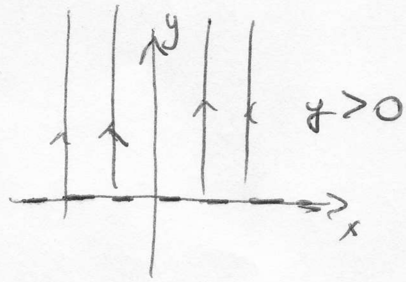
$$\begin{cases} \dfrac{dA}{dt} - AB = 0 \\[2ex] \dfrac{dB}{dt} + A^2 = 0 \end{cases}$$

(1) $A = 0\,,\quad B = \text{const.}$

$$V = AX_1 + BX_2 = \alpha y \partial_y$$

The curve tangent to V:

$$\left.\begin{array}{l} \dfrac{dx}{dt} = 0 \\[2ex] \dfrac{dy}{dt} = \alpha y \end{array}\right\} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix}(t) = \begin{pmatrix} x_0 \\ y_0 e^{\alpha t} \end{pmatrix}$$



(2) $A \neq 0 \Rightarrow B = (\log A)^{\bullet}$

$$\boxed{(\log A)^{\bullet\bullet} + A^2 = 0} \xrightarrow[\substack{\text{equation} \\ C = \log A}]{\text{nice}} \boxed{\ddot{C} + e^{2C} = 0}$$

$$\begin{cases} A = \dfrac{\alpha}{\cosh \alpha(t - t_0)} \\[2ex] B = -\alpha \tanh(\alpha(t - t_0)) \end{cases}$$

$$V = AX_1 + BX_2 = \frac{\alpha}{\cosh \alpha(t-t_0)}\, y(t)\,\partial - \alpha \tanh(\alpha(t-t_0))\, y(t)\, \partial_y$$

The curve:

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{\alpha\, y(t)}{\cosh \alpha(t-t_0)} \\[2ex] \dfrac{dy}{dt} = \left(-\alpha \tanh \alpha(t-t_0)\right) y(t) \Rightarrow \boxed{y(t) = \dfrac{y_0'}{\cosh \alpha(t-t_0)}} \end{cases}$$

$$\frac{dx}{dt} = \frac{\alpha y_0'}{\cosh^2 \alpha (t-t_0)} \Rightarrow x(t) = y_0' \tanh \alpha (t-t_0) + x_0'$$

$$\begin{pmatrix} x \\ y \end{pmatrix}(t) = \begin{pmatrix} y_0' \tanh \alpha (t-t_0) + x_0' \\ \dfrac{y_0'}{\cosh \alpha (t-t_0)} \end{pmatrix}$$

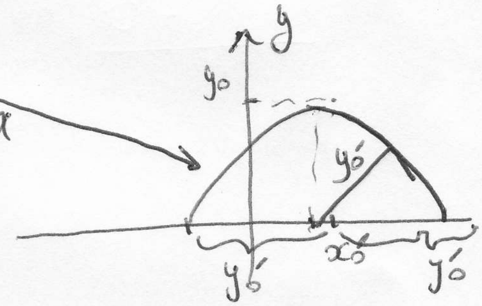
$$\begin{cases} x = y_0' \tanh \alpha (t-t_0) + x_0' \Rightarrow y_0'^2 \tanh^2 \alpha (t-t_0) = (x-x_0')^2 \\ y = \dfrac{y_0'}{\cosh \alpha (t-t_0)} \end{cases}$$

$$y^2 = \frac{y_0'^2}{\cosh^2 \alpha (t-t_0)} = y_0'^2 \, \frac{\cosh^2 \alpha (t-t_0) - \sinh^2 \alpha (t-t_0)}{\cosh^2 \alpha (t-t_0)} =$$

$$= y_0'^2 \left(1 - \tanh^2 \alpha (t-t_0)\right)$$

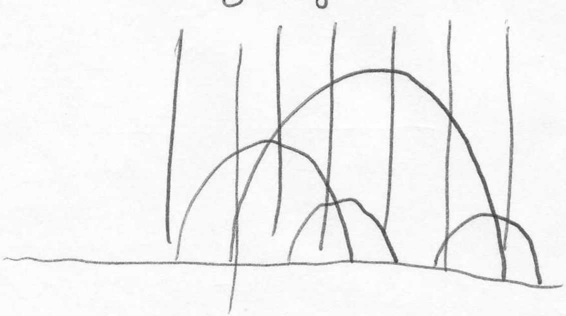
$$\Rightarrow \boxed{y^2 + (x-x_0')^2 = y_0'^2}$$

$y > 0$



semicircles centered at $(x_0', 0)$ of radius y_0'

Two kinds of geodesics:



$$-y_0' \tanh\alpha t_0 + x_0' = x_0$$

$$\frac{y_0'}{\cosh\alpha t_0} = y_0$$

$$y_0' = y_0 \cosh\alpha t_0$$

$$x_0' = x_0 + y_0 \cosh\alpha t_0 \tanh\alpha t_0 =$$

$$= x_0 + y_0 \sinh\alpha t_0$$

$$\begin{cases} x(t) = y_0 \cosh\alpha t_0 \tanh\alpha(t-t_0) + y_0 \sinh\alpha t_0 + x_0 \\ y(t) = \dfrac{y_0 \cosh\alpha t_0}{\cosh\alpha(t-t_0)} \end{cases}$$

$$V_x = \frac{dx}{dt}\Big|_{t=0} = \frac{\alpha y_0}{\cosh\alpha t_0} \Rightarrow \boxed{\cosh\alpha t_0 = \frac{\alpha y_0}{V_x}}$$

$$V_y = \frac{dy}{dt}\Big|_{t=0} = \alpha y_0 \tanh\alpha t_0 \Rightarrow \tanh\alpha t_0 = \frac{v_y}{\alpha y_0}$$

$$\boxed{\sinh\alpha t_0 = \frac{v_y}{v_x}}$$

$$\boxed{x(t) = y_0 \cosh\alpha t_0 \frac{\tanh\alpha t - \tanh\alpha t_0}{1 - \tanh\alpha t \tanh\alpha t_0} + y_0 \frac{v_y\, \alpha y_0}{\alpha y_0\, v_x} + x_0}$$

$$\boxed{= y_0^2 \frac{\alpha}{v_x} \frac{\tanh\alpha t - \frac{v_y}{\alpha y_0}}{1 - \frac{v_y}{\alpha y_0}\tanh\alpha t} + y_0 \frac{v_y}{v_x} + x_0}$$

$$\alpha^2 y_0^2 - v_y^2 = v_x^2$$

$$\boxed{v_y^2 + v_x^2 = \alpha^2 y_0^2}$$

$$\frac{v_y}{\alpha y_0} = \sqrt{1 - \frac{v_x^2}{\alpha^2 y_0^2}}$$

$$\boxed{y(t) = \frac{y_0^2 \frac{\alpha}{v_x}}{\cosh\alpha t \frac{\alpha y_0}{V_x} - \sinh\alpha t \frac{v_y}{v_x}} = \frac{y_0}{\cosh\alpha t - \frac{v_y}{\alpha y_0}\sinh\alpha t}}$$

$$\cancel{\exp_{\binom{x_0}{y_0}}\binom{v_x}{v_y} = \left(y_0^2 \frac{\alpha}{v_x} \frac{\tanh\alpha - \frac{v_y}{\alpha y}}{1 - \frac{v_y}{\alpha y_0}\tanh\alpha} \right.}$$

$$\exp_{\binom{x_0}{y_0}}\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} \dfrac{\sqrt{v_x^2+v_y^2}\, x_0 + (v_x y_0 - v_y x_0)\tanh\frac{\sqrt{v_x^2+v_y^2}}{y_0}}{\sqrt{v_x^2+v_y^2} - v_y \tanh\frac{\sqrt{v_x^2+v_y^2}}{y_0}} \\[2em] \dfrac{y_0\sqrt{v_x^2+v_y^2}}{\sqrt{v_x^2+v_y^2}\cosh\frac{\sqrt{v_x^2+v_y}}{y_0} - v_y \sinh\frac{\sqrt{v_x^2+v_y}}{y_0}} \end{pmatrix}$$

$$\exp_{\binom{x_0}{y_0}}\begin{pmatrix} v_x \\ v_y \end{pmatrix} \simeq \begin{pmatrix} x_0 + v_x + \dfrac{v_x v_y}{y_0} + \dots \\[1em] y_0 + v_y + \dfrac{v_y^2}{2y_0} - \dfrac{v_x^2}{2y_0} + \dots \end{pmatrix}$$